Math 511

Ch 3 "Algebras"

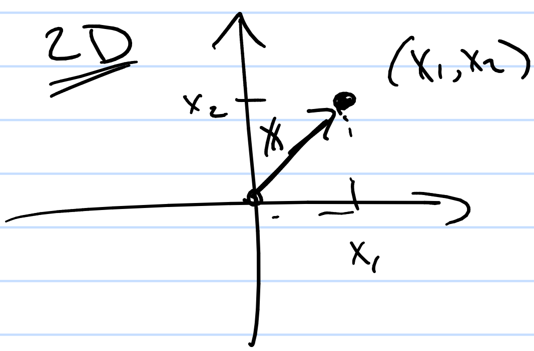
Math(s) = Toys + Rules

Vector Spaces

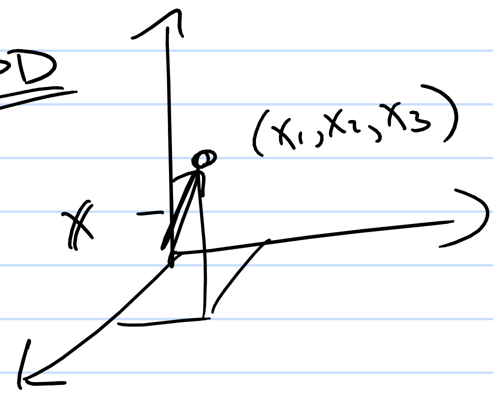
- coordinates

Start w/ real space

2D



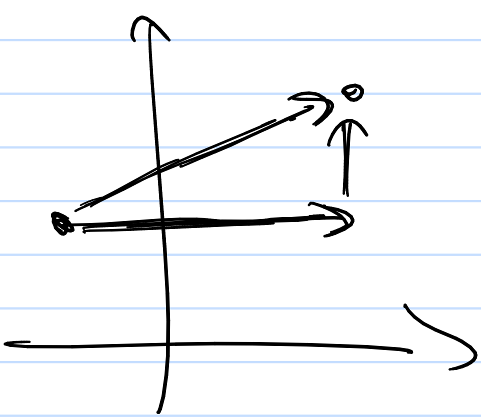
3D

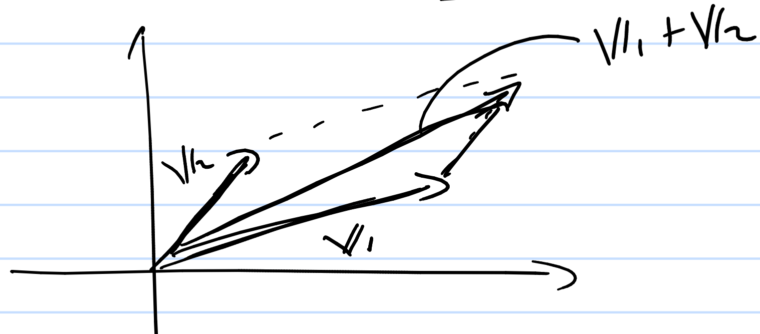


both have vectors

2D $\quad \mathbb{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

3D $\quad \mathbb{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

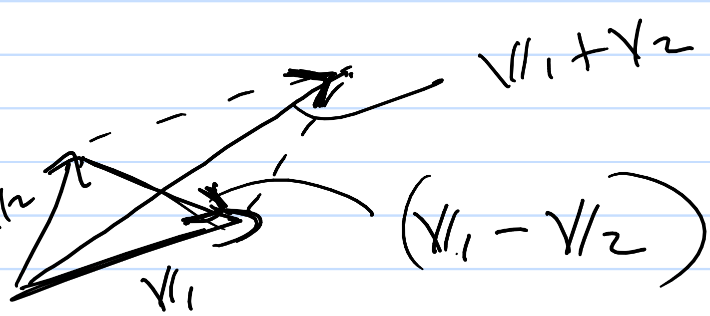


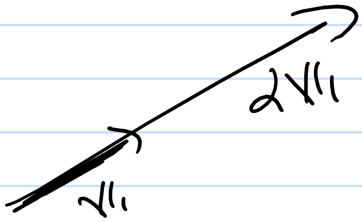


$$\mathbb{V}_1 + \mathbb{V}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

2D

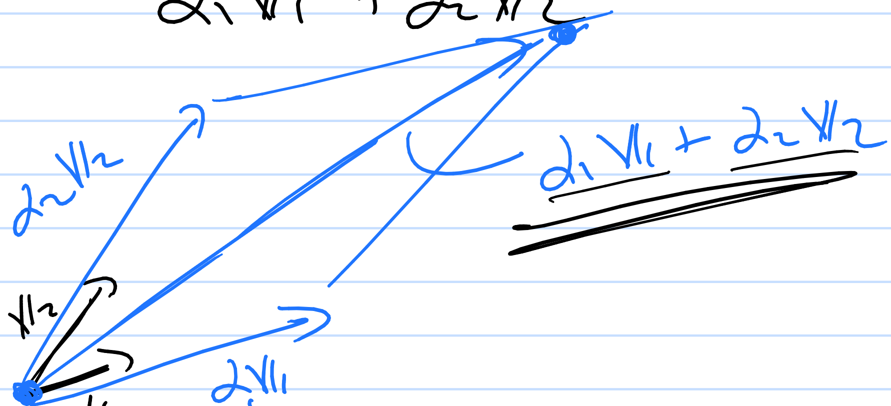
$$= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

① 

② αv_1 

$= \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$

→ <u>linear combination</u> $\alpha_1 v_1 + \alpha_2 v_2$



<u>Toolbox</u> ① <u>Sets</u> = collections of stuff

[ex] $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$A = \{$ all possible matrices $\}$

$B = \{$ all possible polynomials up to degree 3 $\}$

② Inductive Definitions.

(a) Basis Elements of a set.

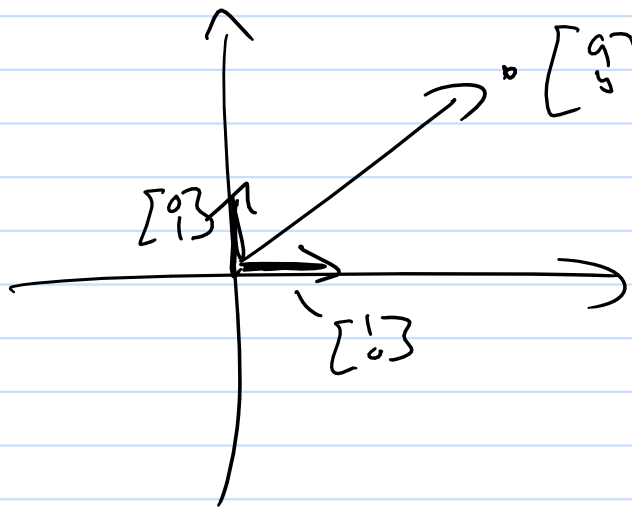
(b) Inductive definition / rule to make new elements from old.

$\mathbb{Z}^+ = \{ |, ||, |||, ||||, |||||, \dots \}$

(a) basis $= \{ | \}$

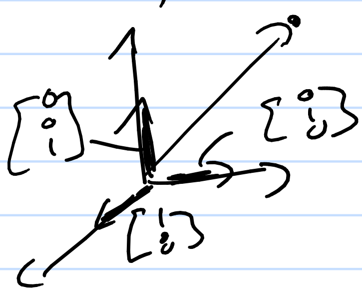
(b) inductive part is get one more.

2D



$$\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$$

3D



$$\begin{bmatrix}a\\b\\c\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$$

---

v_1, v_2 are in the set V. Define αv_1, $v_1 + v_2$

$\boxed{\text{Closure Properties}}$

(1) $v_1 + v_2 \in V$ ✓

(2) $\alpha v_1 \in V$. ✓

Properties for $v_1 + v_2$

A1) $v_1 + v_2 = v_2 + v_1$ ✓

A2) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ ✓

A3) there is a zero vector 0 such that $v_1 + 0 = v_1$ ✓

A4) $v_1 + (-v_1) = 0$ ✓

Properties for $\alpha \mathbf{v}_1$

A5) $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2$

A6) $(\alpha + \beta)\mathbf{v}_1 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_1$

A7) $(\alpha\beta)\mathbf{v}_1 = \alpha(\beta\mathbf{v}_1) = \beta(\alpha\mathbf{v}_1)$

A8) $1 \cdot \mathbf{v}_1 = \mathbf{v}_1$

All 8 axioms hold for

$\boxed{2D}$ w/ $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

and $\alpha\mathbf{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$

and similarly for 3D.

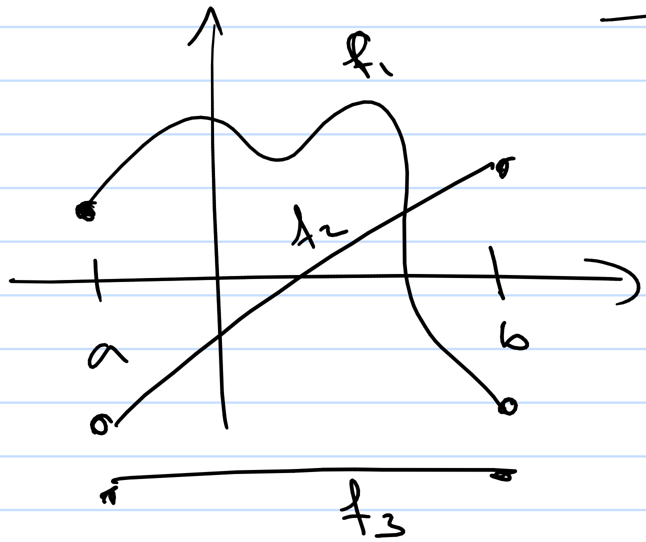
Idea: what if we mess with our set?

And mess with $\mathbf{v}_1 + \mathbf{v}_2$? (binary operator)

And mess with $\alpha\mathbf{v}_1$? (unary operator)

Any Set with $\mathbf{v}_1 + \mathbf{v}_2$, $\alpha\mathbf{v}_1$ defined and all 8 Axioms are true → Call it a Vector Space.

ex) Set = {all cont. functions over $[a,b]$}



Define: $V_1 + V_2$

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$

Define: αV_1

$$(\alpha f_1)(x) = \alpha f_1(x)$$

Is it a vector space?

Check all 8 axioms?

(1) is f_1+f_2 still a cont. function? True
(2) is αf_1 still a cont. function? true

A1) $(f_1+f_2)(x) = f_1(x) + f_2(x)$

$$= f_2(x) + f_1(x)$$

$$= (f_2+f_1)(x)$$

A2) $((f_1+f_2)+f_3)(x) = \ldots\ldots = (f_1+(f_2+f_3))(x)$

??

$Z(x)=0$

A3) 0 exists?

$$(f_1+Z)(x) = f_1(x) + \boxed{Z(x)} = f_1(x)$$