Math 242

Q's

$y = (4x^2 + a)^{1/2}$ two slant asymptotes $y = \pm 2x$

$$\lim_{x\to+\infty} (4x^2+a)^{1/2}$$

$$= \lim_{x\to+\infty} \sqrt{4x^2}\sqrt{1+\frac{a}{4x^2}}$$

$$= \lim_{x\to+\infty} 2|x|\sqrt{1+\cancel{\frac{a}{4x^2}}^{\to 0}}$$

$$= \lim_{x\to+\infty} 2|x|$$

slant asym $y = \pm 2x$ ($x\to+\infty$, $x\to-\infty$)

$$y = \sqrt{4x^4+a} = \sqrt{4x^4}\sqrt{1+\cancel{\frac{a}{4x^4}}^{\to 0}}$$

$$\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty} 2x^2$$

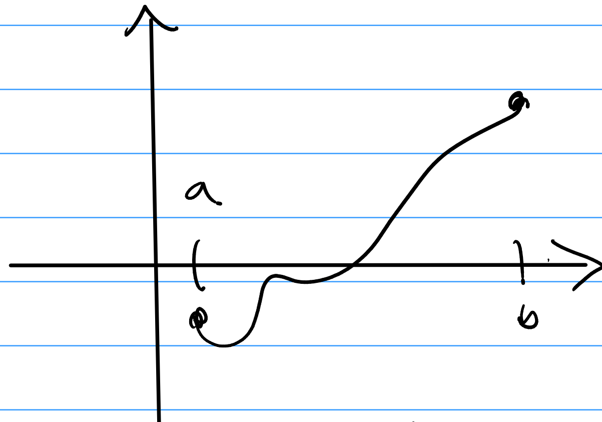
$$y = \frac{x^2+1}{x+1} \quad \text{(asymptotes)}$$

$$\lim_{x\to+\infty}\frac{x^2+1}{x+1} = \lim_{x\to+\infty}\frac{x^2(1+1/x^2)}{x(1+1/x)}$$

$$= \lim_{x\to+\infty} x\,\frac{(1+1/x^2)}{(1+1/x)} = \lim_{x\to+\infty} x = +\infty$$

Application: Newton's Method

Finding roots



bisection method

① $f(a) \cdot f(b) < 0$ $f(a), f(b)$ are on opposite sides of x-axis.

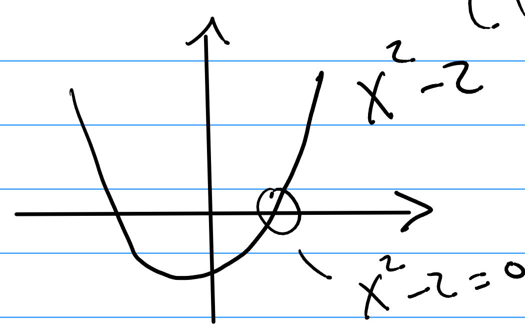
② let $c = \frac{a+b}{2}$

→ check $f(a) \cdot f(c) < 0$?
$f(c) \cdot f(b) < 0$?   whichever is yes → new $[a, b]$

(loop)

Problem: slow

ex $\sqrt{2} \approx$?



$f(x) = x^2 - 2$

$f(1) = 1^2 - 2 = -1$
$f(2) = 2^2 - 2 = 2$ opposite sides

→ $\frac{1+2}{2} = 1.5$ $f(1.5) = 0.25$

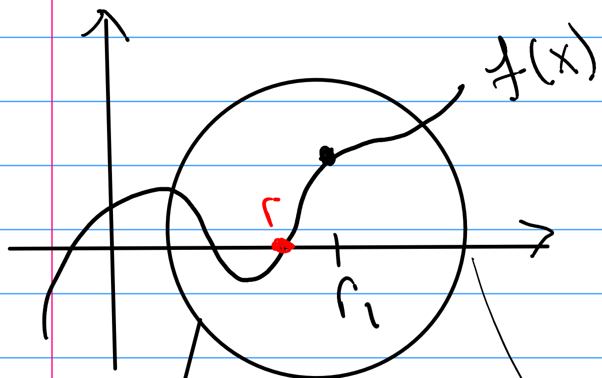
between 1 and 1.5

$f(1.25) =$

Faster way?

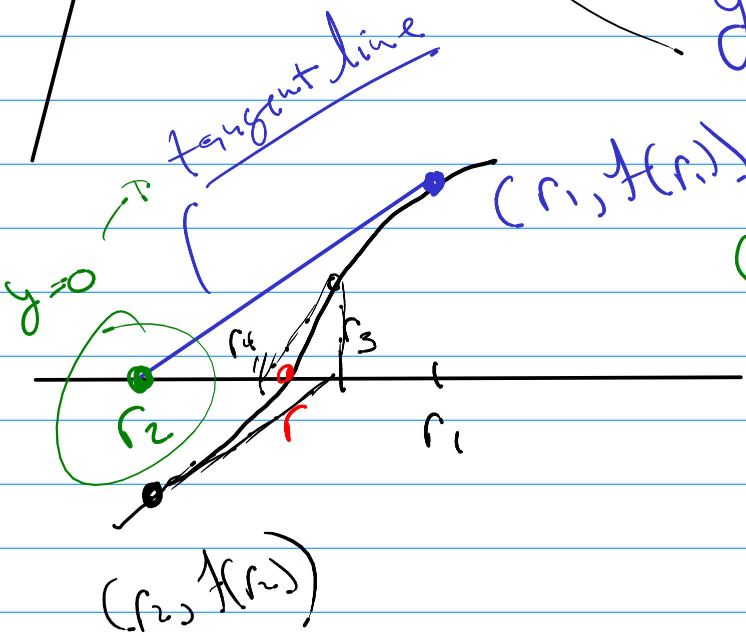
Newton's Method

$r_1 = 1^{st}$ guess at root r



tangent line eqn

$$y - f(r_1) = f'(r_1)(x - r_1)$$



tangent line root

$$0 - f(r_1) = f'(r_1)(r_2 - r_1)$$

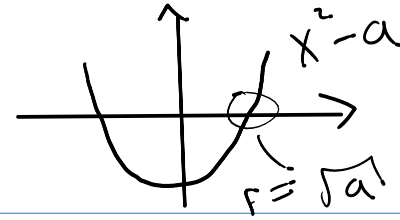
$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

Newton's Method Algorithm

$$\boxed{\begin{aligned} r_1 &= \text{good guess} \\ r_{n+1} &= r_n - \frac{f(r_n)}{f'(r_n)} \end{aligned}}$$

$$\lim_{n \to +\infty} r_n = r$$

ex $\sqrt{a} \approx ?$



$$f(x) = x^2 - a \quad , \quad f'(x) = 2x$$

Newton's Method $\quad r_1 = \text{good guess}$

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)} \longrightarrow r_{n+1} = r_n - \frac{r_n^2 - a}{2r_n}$$

$$r_{n+1} = r_n - \left(\frac{r_n^2}{2r_n} - \frac{a}{2r_n}\right)$$

$$r_{n+1} = r_n - \frac{1}{2} r_n + \frac{a}{2r_n}$$

$$r_{n+1} = \frac{1}{2} r_n + \frac{a}{2r_n}$$

$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{a}{r_n}\right)$$

$\sqrt{a}$ Algorithm $\quad r_1 = \text{good guess}$

$$r_{n+1} = (r_n + a/r_n)/2$$

So $\sqrt{2}$ $\quad r_1 = 1$

$$r_{n+1} = (r_n + 2/r_n)/2$$

So $\sqrt{3}$ $\quad r_1 = 1$

$$r_{n+1} = (r_n + 3/r_n)/2$$

Newton's Method

$r_1 =$ good guess ← stay away from $f'(x) = 0$

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$

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ex

buy a car

Now: \$18,000

"Later": 60 payments of \$375 → (\$22,500)

↑
5 yrs

$$\text{Present Value} = \text{Payment}\left[\frac{1-(1+r)^{-n}}{r}\right]$$

$n \equiv$ number of payments
$r =$ interest rate

ex

$$18{,}000 = 375\left[\frac{1-(1+r)^{-60}}{r}\right]$$

$$48 = \frac{1-(1+r)^{-60}}{r}$$

$$48r = 1-(1+r)^{-60}$$

$$48r(1+r)^{60} = (1+r)^{60} - 1$$

$$48r(1+r)^{60} - (1+r)^{60} + 1 = 0$$

find root?!

Newton's Method

$$f(r) = 48r(1+r)^{60} - (1+r)^{60} + 1$$

$$f'(r) = 12(1+r)^{59}(244r - 1)$$

$$r_1 = 0.01$$

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$

$$r_{n+1} = r_n - \frac{48r_n(1+r_n)^{60} - (1+r_n)^{60} + 1}{12(1+r_n)^{59}(244r_n - 1)}$$

Antiderivatives

$$D_x[f(x)] = f'(x)$$

How to undo derivatives?

Know derivatives really really well.

Def $F(x)$ is $\boxed{\text{an}}$ antiderivative of $f(x)$ on an interval I if $D_x[F(x)] = f(x)$ for all $x \in I$.

ex) $\frac{1}{3}x^3 + \pi$ is an antiderivative of x^2

$\boxed{b/c}$ $D_x\left[\frac{1}{3}x^3 + \pi\right] = x^2 + 0 = x^2$

$\boxed{\text{Th}^m}$ if $F(x)$ is an antiderivative of $f(x)$

then $F(x) + C$, C is an arbitrary constant

is the most general antiderivative.

Find antiderivatives?

Know: $D_x[F(x) + C] = f(x)$

rewrite: $A_x[f(x)] = F(x) + C$

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b/c i) $D_x[\sin(x)] = \cos x$

$A_x[\cos(x)] = \sin x + C$

ii) $D_x\{\tan(x)\} = \sec^2 x$

$A_x[\sec^2(x)] = \tan x + C$

$$D_x[x^n] = nx^{n-1}$$

$$A_x[nx^{n-1}] = x^n + C$$

$$A_x[3x^2] = x^3 + C$$

b/c $D_x[x^3 + c] = 3x^{3-1}$

power in front, take 1 from power = new power

$$D_x[x^3] = 3x^2$$

$$A_x[3x^2] = \frac{3}{3}x^{2+1} = x^3$$

→ $A_x[x^n] = \frac{1}{n+1}x^{n+1} + C$

b/c $D_x\left[\frac{1}{n+1}x^{n+1} + C\right] = x^n$

$$A_x[\cos x - \csc x \cot x + x^3]$$

$$= \sin x + \csc x + \frac{1}{4}x^4 + C$$