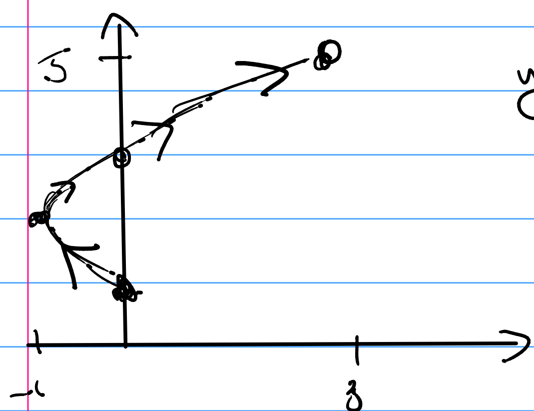
Math 243

Exam 3

(1) $x = t^2 - 2t$ $\quad t = y-1 \quad$ $y = t+1$ $\quad \boxed{0 \le t \le 4}$

$x = (y-1)^2 - 2(y-1)$

$x = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3 = (y-1)(y-3)$



$y = 2 \Rightarrow x = -1$

t	x	y
0	0	1
1	-1	2
4	8	5

(2) $x = t^2 - 2t \qquad y = t+1$

$\frac{dy}{dx} = \frac{y'}{x'} = \boxed{\frac{1}{2t-2}} = (2t-2)^{-1}$

$\frac{d^2y}{dx^2} = \frac{\left(\frac{1}{2t-2}\right)'}{x'} = \frac{-2(2t-2)^{-2}}{2t-2} = \boxed{\frac{-2}{(2t-2)^3}}$

(3) $x = t^2 \qquad y = t^3 - 3t$

$\frac{dy}{dx} = \frac{y'}{x'} = \frac{3t^2-3}{2t} = \frac{3}{2}\,\frac{(t+1)(t-1)}{t}$

horz @ $t=\pm 1$ pts $t=1 \quad (1,-2)$

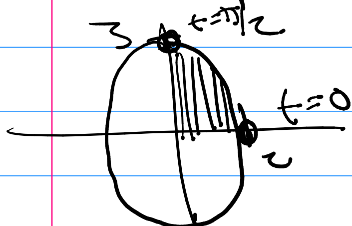
$t=-1 \quad (1,2)$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{3}{2}\frac{t^2-1}{t}\right)'}{2t} = \frac{3}{4}\frac{\left(t-\frac{1}{t}\right)'}{t} = \frac{3}{4}\frac{\left(1+\frac{1}{t^2}\right)}{t}$$

$$= \frac{3}{4}\left(\frac{t^2+1}{t^3}\right)$$

down 0 up

④ ellipse $x = 2\cos t \quad y = 3\sin t$



$$4\int_{\pi/2}^{0} y\,dx = -4\int_{\pi/2}^{0} 3\sin t\, 2\sin t\,dt$$

$$= 24\int_0^{\pi/2} \boxed{\sin^2 t}\,dt$$

$\cos 2t = 1 - 2\sin^2 t$

$\sin^2 t = \frac{1}{2}(1-\cos 2t)$ $\quad = 12\int_0^{\pi/2}(1-\cos 2t)\,dt$

$$= 12\left(t - \cancel{\tfrac{1}{2}\sin 2t}\right)\Big|_0^{\pi/2} = 12\,\pi/2 = \boxed{6\pi}$$

⑤ $x = 1+t^2 \quad y = 1+t^3 \qquad t\in[0,1]$

$$AL = \int_0^1\sqrt{(x')^2+(y')^2}\,dt = \int_0^1\sqrt{(2t)^2+(3t^2)^2}\,dt$$

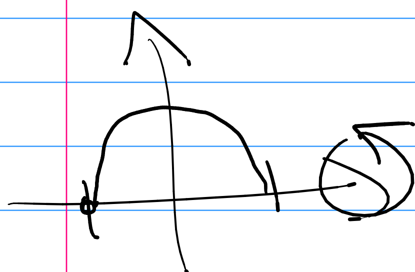
$$= \int_0^1\sqrt{4t^2+9t^4}\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt$$

let $u = 4+9t^2$

$du = 18t\,dt \qquad \frac{1}{18}\int_4^{13} u^{1/2}\,du$

$= \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big|_4^{13} = \boxed{\frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)}$

6) $x = r\cos t \quad y = r\sin t \qquad t \in [0, \pi]$



$SA = \int_0^{\pi} 2\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$

$= \int_0^{\pi} 2\pi r^2 \sin t\,dt = -2\pi r^2\cos t\Big|_0^{\pi}$

$= \boxed{4\pi r^2}$

7) a) $(2, \pi/3)$
$r \quad \theta$

$x = \boxed{2\cos \pi/3}$

$y = \boxed{2\sin \pi/3}$

b) $(1, 1)$ $\qquad r^2 = 2 \rightarrow r = \pm\sqrt{2}$
x, y $\qquad \tan\theta = 1 \rightarrow \pi/4 + n\pi$

$(\sqrt{2}, \pi/4)$

$(-\sqrt{2}, 5\pi/4)$

8) $r = \cos\theta \qquad x = \cos^2\theta$

$$y = \cos\theta \sin\theta$$

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{-\sin^2\theta + \cos^2\theta}{2\cos\theta\sin\theta} = \frac{\cos 2\theta}{\sin 2\theta}$$

<u>horz:</u> $\cos 2\theta = 0 \rightarrow$ $\theta = \pi/4 + n\pi/2$

vert $\sin 2\theta = 0 \rightarrow$ $\theta = 0 + n\pi/2$

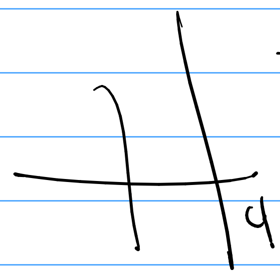
a) $r = 4\sec\theta \qquad x = r\cos\theta$

$y = r\sin\theta$

$r = \dfrac{4}{\cos\theta} \qquad r^2 = x^2 + y^2$

$r\cos\theta = 4 \qquad \tan\theta = y/x$

$\boxed{x = 4}$



10) $A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta \qquad r = \sqrt{\ln\theta}$

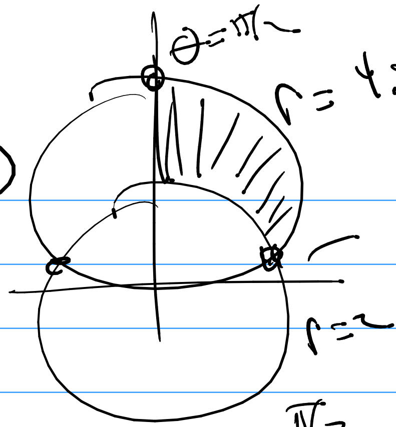
$$A = \frac{1}{2}\int_1^{2\pi} \ln\theta \, d\theta = \frac{1}{2}\left[\theta\ln\theta - \int d\theta\right]\Big|_1^{2\pi}$$

$$f(\theta) = \ln\theta \rightarrow f'(\theta) = \frac{1}{\theta}$$

$$g'(\theta) = 1 \rightarrow g(\theta) = \theta$$

$$A = \frac{1}{2}\left[\theta\ln\theta - \theta\right]_1^{2\pi} = \text{etc}$$

(11)



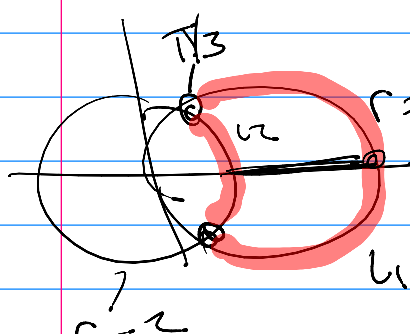
$2 = 4\sin\theta \qquad \tfrac{1}{2} = \sin\theta \qquad \theta = \pi/6$

$$2\int_{\pi/6}^{\pi/2} \tfrac{1}{2}(4\sin\theta)^2 - \tfrac{1}{2}(2)^2 \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 16\sin^2\theta - 4 \, d\theta = \text{[illegible]}$$

(12) $r = 4\cos\theta \qquad r = 2$

$$AL = \int_\alpha^\beta \sqrt{r^2 + (r')^2}\, d\theta$$



$\cos\theta = \tfrac{1}{2}$

$$L_1 = 2\int_0^{\pi/3} \sqrt{16\cos^2\theta + 16\sin^2\theta}\, d\theta$$

$$= 2\int_0^{\pi/3} 4\, d\theta = 8\,\pi/3 = \tfrac{8}{3}\pi$$

$$L_2 = 2\int_0^{\pi/3} \sqrt{4}\, d\theta = 4\int_0^{\pi/3} d\theta = \tfrac{4}{3}\pi$$

$$\boxed{\tfrac{8}{3}\pi + \tfrac{4}{3}\pi}$$

ellipse $c^2 = a^2 - b^2$

(13) $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ hyper $c^2 = a^2 + b^2$

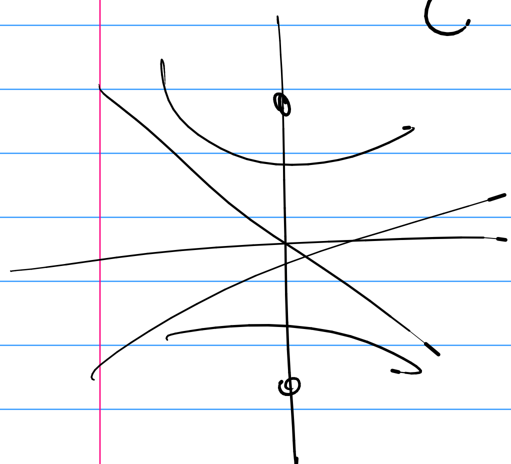
$9(x^2 - 8x + 16) - 4(y^2 - 2y + 1) = -176 + 144 - 4$

$$\frac{9(x-4)^2}{-36} - \frac{4(y-1)^2}{-36} = \frac{-36}{-36}$$

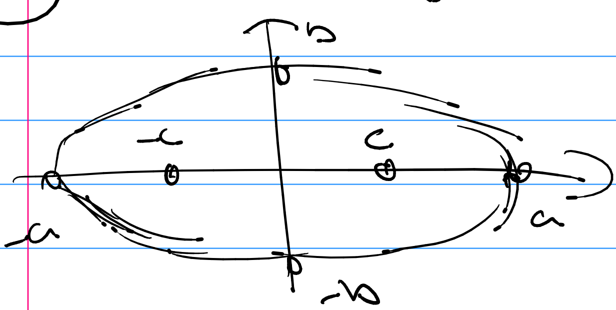
$$\frac{(y-1)^2}{3^2} - \frac{(x-4)^2}{2^2} = 1 \quad \text{hyperbola}$$

$c^2 = a^2 + b^2 = 3^2 + 2^2 = 13$

$(0, \pm\sqrt{13})$



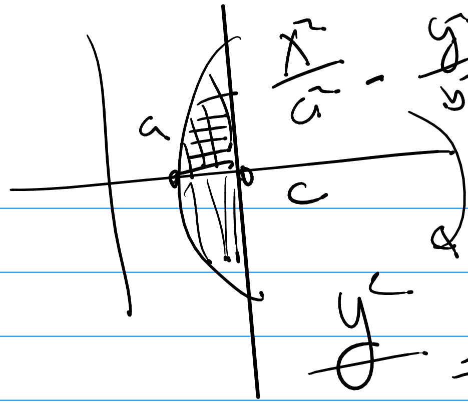
(14) foci $(\pm 2, 0)$ vertices $(\pm 5, 0)$



$4 = 25 - b^2$

$$\boxed{\frac{x^2}{25} + \frac{y^2}{21} = 1}$$

15 Hardest



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\boxed{c^2 = a^2 + b^2}$

$A = 2\int_a^c y\,dx$

$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$

$y = b\sqrt{\left(\frac{x}{a}\right)^2 - 1}$

$A = \boxed{2b\int_a^c \sqrt{\left(\frac{x}{a}\right)^2 - 1}\,dx}$

let $u = x/a$

$du = \frac{1}{a}dx$

$A = 2ab\int_1^{c/a} \sqrt{u^2-1}\,du$

(★) $\int \sqrt{u^2-1}\,du = \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\ln\left(u + \sqrt{u^2-1}\right)$

Note! $\sqrt{\frac{c^2}{a^2} - 1} = \sqrt{\frac{c^2-a^2}{a^2}} = \sqrt{\frac{b^2}{a^2}} = \frac{b}{a}$

$= \cdots = \boxed{\frac{b^2c}{a} - ab\ln\left(\frac{b+c}{a}\right)}$

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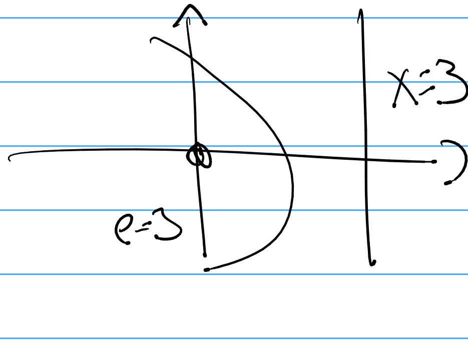

$x = 3 = d$

$r = \frac{ed}{1 + e\cos\theta}$

$\boxed{r = \frac{9}{1 + 3\cos\theta}}$

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$$r = \frac{ed}{\boxed{1} \pm e\cos\theta}$$

$$r = \frac{5}{\boxed{2} - 4\cos\theta}$$

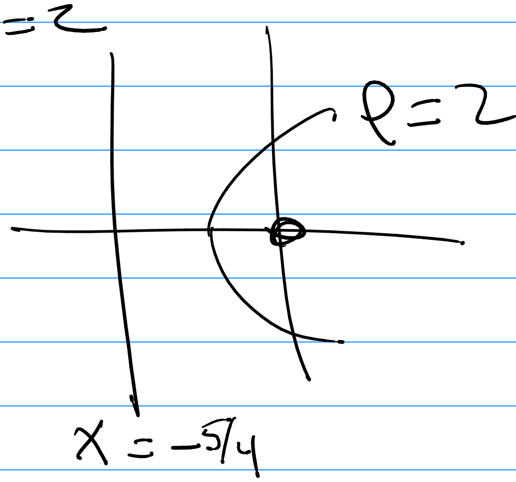
$$r = \frac{5/2}{1 - 2\cos\theta}$$

$ed = 5/2 \rightarrow 2d = 5/2$

$d = 5/4$

$e = 2$

$$r = \frac{ed}{1 - e\cos\theta}$$



hyperbola